THE COSMIC EXPLORER GRAVITATIONAL WAVE OBSERVATORY

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Signal Recycling Cavity Gouy Phase and Mode Matching

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1 Introduction

The most critical thermal lensing in a Dual-Recycled Fabry-Perot Michelson Interferometer (DRFPMI) happens at the input test masses (ITM). The high arm optical power leads to a significant heat deposition on the ITM, which induces a thermal lens in the ITM substrate, which affects the recycling cavities.

This note asks the question how the mode matching between the arm eigenmode and the Signal Recycling Cavity (SRC, or Signal Extraction Cavity SEC - old habits die hard) Eigenmode is affected by this thermal lens. It shows that the SRC Gouy phase has a huge impact on this coupling.

Analytical approaches are used:

1. Using Ray Transfer Matrices we first show that for 45 deg one-way SRC Gouy phase the mode matching is first-order insensitive to (pure parabolic) lenses in the ITM.
2. Next, using a perturbation theory approach, we show that a similar condition holds for higher order modes.
3. Finally, we estimate the coupling for a theoretical thin substrate thermal lens.

2 Limitations

Since we try to understand the basic dependence using analytical methods to guide future simulations, we make some simplifications.

- We assume only the recycling cavity eigenmode is changes, with the arm mode staying the same. The thermorefractive ITM lens is indeed bigger than the effective radius of curvature change in the ITM, but a full analysis will also have to look at the arm cavity change.
- We only look at small thermal lenses, i.e. we look at the linear deviation from perfect matching due to a thermal lens.

3 Recycling Cavity Representation with Flat Retro-Reflection

The optical elements of the SRC are a series of lenses or curved mirrors (L), alternating with free space propagation (D). Starting just before the ITM bulk lens, the round-trip telescope is given by

$$\begin{aligned} M^{\mathrm{SRC}}_{\mathrm{RT,\,starting\,before\,ITM\,bulk}} = \; & L_{\mathrm{ITM\,back}} \\ & \cdot \left(L_{\mathrm{ITM\,bulk}} D_{\mathrm{ITM} \leftarrow \mathrm{M3}} L_{\mathrm{M3}} D_{\mathrm{M3} \leftarrow \mathrm{M2}} L_{\mathrm{M2}} D_{\mathrm{M2} \leftarrow \mathrm{SRM}}\right) \\ & \cdot \; L_{\mathrm{SRM}} \\ & \cdot \left(D_{\mathrm{SRM} \leftarrow \mathrm{M2}} L_{\mathrm{M2}} D_{\mathrm{M2} \leftarrow \mathrm{M3}} L_{\mathrm{M3}} D_{\mathrm{M3} \leftarrow \mathrm{ITM}} L_{\mathrm{ITM\,bulk}}\right) \end{aligned} \tag{1}$$

For the purpose of this note we will enforce a "flat" termination mirror, i.e. we include exactly one half of the SRM and ITM lenses at the beginning and end of the telescope:

$$\begin{aligned} M^{\mathrm{SRC}}_{\mathrm{RT}} = & \left(L_{\frac{1}{2}\,\mathrm{ITM\,back}} L_{\mathrm{ITM\,bulk}} D_{\mathrm{ITM} \leftarrow \mathrm{M3}} L_{\mathrm{M3}} D_{\mathrm{M3} \leftarrow \mathrm{M2}} L_{\mathrm{M2}} D_{\mathrm{M2} \leftarrow \mathrm{SRM}} L_{\frac{1}{2}\,\mathrm{SRM}}\right) \\ & \cdot \left(L_{\frac{1}{2}\,\mathrm{SRM}} D_{\mathrm{SRM} \leftarrow \mathrm{M2}} L_{\mathrm{M2}} D_{\mathrm{M2} \leftarrow \mathrm{M3}} L_{\mathrm{M3}} D_{\mathrm{M3} \leftarrow \mathrm{ITM}} L_{\mathrm{ITM\,bulk}} L_{\frac{1}{2}\,\mathrm{ITM\,back}}\right) \\ & = M^{\mathrm{SRC\,reverse}}_{\mathrm{1way}} M^{\mathrm{SRC}}_{\mathrm{1way}} \end{aligned} \tag{2}$$

Here we have

$$\begin{aligned} L_{\mathrm{ITM\,back}} &= \left(L_{\frac{1}{2}\,\mathrm{ITM\,back}}\right)^2 \\ L_{\mathrm{SRM}} &= \left(L_{\frac{1}{2}\,\mathrm{SRM}}\right)^2 . \end{aligned} \tag{3}$$

4 Ray Transfer Matrix Approach

4.1 Setup

The one-way, flat propagation matrix is given by

$$M^{\mathrm{SRC}}_{\mathrm{1way}} = \left(L_{\frac{1}{2}\,\mathrm{SRM}} D_{\mathrm{SRM} \leftarrow \mathrm{M2}} L_{\mathrm{M2}} D_{\mathrm{M2} \leftarrow \mathrm{M3}} L_{\mathrm{M3}} D_{\mathrm{M3} \leftarrow \mathrm{ITM}} L_{\mathrm{ITM\,bulk}} L_{\frac{1}{2}\,\mathrm{ITM\,back}}\right) \tag{4}$$

where we now represent the elements with Ray Transfer Matrices

$$D_d = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \tag{5}$$

and

$$L_f = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\kappa & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{pmatrix} \tag{6}$$

for a distance d or focal length $f = \frac{1}{\kappa} = \frac{R}{2}$.

We can write this one-way propagation matrix as

$$M^{\mathrm{SRC}}_{\mathrm{1way}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{7}$$

with $\det(M^{\mathrm{SRC}}_{\mathrm{1way}}) = ad - bc = 1$.

Also, with

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{8}$$

the reverse propagation matrix is given by

$$M_{\text{1way}}^{\text{SRC reverse}} = \sigma_3 \left(M_{\text{1way}}^{\text{SRC}}\right)^{-1} \sigma_3 = \begin{pmatrix} d & b \\ c & a \end{pmatrix} \tag{9}$$

and

$$M_{\text{RT}}^{\text{SRC}} = \begin{pmatrix} ad + bc & 2bd \\ 2ac & ad + bc \end{pmatrix}. \tag{10}$$

4.2 Characteristic Polynomial and Gouy Phase

Since $\det(M_{\text{RT}}^{\text{SRC}}) = 1$, the characteristic polynomial determining the eigenvalues λ is given by

$$\lambda^2 - 2(ad + bc)\lambda + 1 = 0. \tag{11}$$

Assuming the cavity is stable (i.e. $-1 < ad + bc < 1$) we have

$$\lambda^2 - 2\cos(\phi_{\text{RT}})\lambda + 1 = 0 \tag{12}$$

with the eigenvalues

$$\lambda = \exp(\pm i\phi_{\text{RT}}) = \exp(\pm 2i\phi_1) \tag{13}$$

where we also introduced the one-way Gouy phase $\phi_1 = \phi_{\text{RT}}/2$. Note that we have

$$\cos(\phi_{\text{RT}}) = ad + bc. \tag{14}$$

and

$$\sin^2(\phi_{\text{RT}}) = -4adbc. \tag{15}$$

We find that the Rayleigh range z_R, which is defined to be positive, is given by

$$z_R^2 = -\frac{bd}{ac}. \tag{16}$$

We also get the identities

$$2bd = z_R \sin\phi_{\text{RT}} \tag{17}$$

and

$$2ac = -\frac{\sin\phi_{\text{RT}}}{z_R}. \tag{18}$$

Thus, the rond trip ray transfer matrix can be parametrized as

$$M_{\text{RT}}^{\text{SRC}} = \begin{pmatrix} ad + bc & 2bd \\ 2ac & ad + bc \end{pmatrix} = \begin{pmatrix} \cos\phi_{\text{RT}} & z_R \sin\phi_{\text{RT}} \\ -\frac{\sin\phi_{\text{RT}}}{z_R} & \cos\phi_{\text{RT}} \end{pmatrix} \tag{19}$$

which has the eigenvector and value

$$M_{\text{RT}}^{\text{SRC}} \begin{pmatrix} iz_R \\ 1 \end{pmatrix} = \exp(-i\phi_{\text{RT}}) \begin{pmatrix} iz_R \\ 1 \end{pmatrix}, \tag{20}$$

as well as their complex conjugate. This sign convention is consistent with the imaginary part of $q = z + iz_R$ being positive, i.e. positive Rayleigh range. Note that the real part z of the Gaussian beam parameter $q = z + iz_R$ is zero because of the "flat" condition we imposed on the cavity.

4.3 Thermal Lens

We can now introduce an additional thermal lens T_κ with κ diopter in the ITM:

$$T_\kappa = \begin{pmatrix} 1 & 0 \\ -\kappa & 1 \end{pmatrix} \tag{21}$$

The new round-trip matrix is then

$$M_\kappa^{\rm SRC} = T_\kappa M_{\rm RT}^{\rm SRC} T_\kappa = \begin{pmatrix} (a-b\kappa)d + b(c-d\kappa) & 2bd \\ 2(a-b\kappa)(c-d\kappa) & (a-b\kappa)d + b(c-d\kappa) \end{pmatrix}. \tag{22}$$

With $a(\kappa) = (a - b\kappa)$ and $c(\kappa) = (c - d\kappa)$ we can write this in the old form

$$M_\kappa^{\rm SRC} = \begin{pmatrix} a(\kappa)d + bc(\kappa) & 2bd \\ 2a(\kappa)c(\kappa) & a(\kappa)d + bc(\kappa) \end{pmatrix}. \tag{23}$$

We now find the eigenvalues and vectors as

$$M_\kappa^{\rm SRC} \begin{pmatrix} iz_R(\kappa) \\ 1 \end{pmatrix} = \exp(-i\phi_{\rm RT}(\kappa)) \begin{pmatrix} iz_R(\kappa) \\ 1 \end{pmatrix} \tag{24}$$

Note that since we defined the cavity as "flat", the real part of $q = z + iz_R$ is guaranteed to be zero even with a thermal lens, and we have

$$q(\kappa) = iz_R(\kappa) = i\sqrt{-\frac{bd}{a(\kappa)c(\kappa)}} = -i\frac{\sin(\phi_{\rm RT}(\kappa))}{2a(\kappa)c(\kappa)}. \tag{25}$$

Thus we can find the beam size w on thee ITM from

$$\frac{\lambda}{\pi w^2} = -{\rm imag}\frac{1}{q} = \frac{1}{z_R} \tag{26}$$

or simply

$$w(\kappa)^4 = \frac{\lambda^2}{\pi^2} z_R^2(\kappa) = -\frac{\lambda^2}{\pi^2}\frac{bd}{a(\kappa)c(\kappa)}. \tag{27}$$

Now we want to calculate the relative change in spot size w, where the prime $'$ denoted $d/d\kappa$:

$$\frac{w'}{w}|_{\kappa=0} = \log' w|_{\kappa=0} = \frac{1}{4}\log' w^4|_{\kappa=0} = -\frac{1}{4}\log'(-a(\kappa)c(\kappa))|_{\kappa=0} = \frac{ad+bc}{4ac} = -\frac{\cos\phi_{\rm RT}}{\sin\phi_{\rm RT}}\frac{z_R}{2} \tag{28}$$

In words, the change in the mode matching parameter w'/w is given to first order by

$$\frac{w'}{w}|_{\kappa=0} = \frac{ad+bc}{4ac} = -\frac{\cos\phi_{\rm RT}}{\sin\phi_{\rm RT}}\frac{z_R}{2} \tag{29}$$

Alternatively, we can expand the mode overlap of two Gaussian beams

$$\left\langle \mathbf{\Psi}_q | \mathbf{\Psi}_{q(\kappa)} \right\rangle = \frac{2i\sqrt{z_R z_R(\kappa)}}{q - q'^*(\kappa)} = \frac{2\sqrt{z_R z_R(\kappa)}}{z_R(\kappa) + z_R} = \frac{2}{\sqrt{\frac{z_R(\kappa)}{z_R}} + \sqrt{\frac{z_R}{z_R(\kappa)}}} = \frac{2}{\frac{w(\kappa)}{w} + \frac{w}{w(\kappa)}} \tag{30}$$

which to 2nd order in κ is

$$\left\langle \mathbf{\Psi}_q | \mathbf{\Psi}_{q(\kappa)} \right\rangle = \frac{2}{\frac{w(\kappa)}{w} + \frac{w}{w(\kappa)}} \approx 1 - \frac{1}{2}\left(\frac{w'}{w}|_{\kappa=0}\right)^2 \kappa^2 = 1 - \frac{1}{2}\left(-\frac{\cos\phi_{\rm RT}}{\sin\phi_{\rm RT}}\frac{z_R}{2}\right)^2 \kappa^2 \tag{31}$$

4.4 Mode-Matching at SRM

Similarly, we can ask what the impact of the same ITM thermal lens is on mode-matching at the SRM. We can repeat the same analysis, except looking at the beam size $\tilde{w}$ at the SRM. We find:

$$\frac{\tilde{w}'}{\tilde{w}}|_{\kappa=0} = \frac{1}{4ac} = -\frac{1}{\sin\phi_{\rm RT}}\frac{z_R}{2} \tag{32}$$

4.5 Interpretation

The equation

$$\frac{w'}{w}|_{\kappa=0} = -\frac{\cos\phi_{\rm RT}}{\sin\phi_{\rm RT}}\frac{z_R}{2} \tag{33}$$

suggest that SRC and arm remain mode-matched for a round trip Gouy phase of 90 deg (one-way Gouy phase of 45 deg). Note that this corresponds to 180 degree of phase evolution for the relevant LG10 mode.

Thus it seems reasonable to ask whether a similar equation hold for higher-order mode mismatch. This inspired the next calculation.

5 Perturbation Theory Approach

5.1 Mode-Matching at ITM

We start with the same telescope, but now set up an orthonormal basis of Gaussian beams (e.g. HG of LG modes - I use HG mode indices n and m for simplicity), such that the fundamental mode is the eigenmode of the unperturbed SRC cavity:

$$M_0|00> = \exp(-i\phi_{\rm RT})|00> = \lambda_0|00> \tag{34}$$

where we introduced $\lambda_0 = \exp(-i\phi_{\rm RT})$ to simplify notation. For higher-order modes the round trip propagation equation gives

$$M_0|nm> = \lambda_0^{n+m+1}|mn> \tag{35}$$

We now introduce a generalized thermal lens ϵT, where we explicitly define a small, dimensionless parameter. The cavity round trip operator therefore is

$$M = (1+\epsilon T)M_0(1+\epsilon T) = M_0 + \epsilon(TM_0 + M_0T). \tag{36}$$

We can also expand the perturbed state and eigenvalues

$$|\Psi> = |00> + \epsilon|v_1> \tag{37}$$

$$\lambda = \lambda_0 + \epsilon\lambda_1 \tag{38}$$

Note that we can assume

$$< 00|v_1 >= 0 \;\; , \;\; < 00|00 >= 1 \;\text{ and }\; < \Psi|\Psi >= 1 \tag{39}$$

The new cavity eigenmode equation is

$$M|\Psi >= \lambda|\Psi > \tag{40}$$

Explicitly, this is

$$(M_0 + \epsilon T M_0 + \epsilon M_0 T))|(|00 > +\epsilon|v_1 >) = (\lambda_0 + \epsilon\lambda_1)(|00 > +\epsilon|v_1 >) \tag{41}$$

The zeroth order of this equation is trivially true. The first order requires

$$(TM_0 + M_0 T)|00 > +M_0|v_1 >= \lambda_1|00 > +\lambda_0|v_1 > \tag{42}$$

Bracketing with $< 00|$ gives

$$(\lambda_0 + \lambda_0) < 00|T|00 >= \lambda_1 \tag{43}$$

and bracketing with $< mn|$ (not equal to $< 00|$) gives

$$(\lambda_0 + \lambda_0^{n+m+1}) < mn|T|00 > +\lambda_0^{n+m+1} < mn|v_1 >= \lambda_0 < mn|v_1 > \tag{44}$$

which we can solve for the coefficients of $|v_1 >$:

$$< mn|v_1 >= \frac{\lambda_0 + \lambda_0^{n+m+1}}{\lambda_0 - \lambda_0^{n+m+1}} < mn|T|00 >= \frac{\lambda_0^{-(n+m)/2} + \lambda_0^{(n+m)/2}}{\lambda_0^{-(n+m)/2} - \lambda_0^{(n+m)/2}} < mn|T|00 > \tag{45}$$

which can be written as

$$< mn|v_1 >= -i\frac{\cos\frac{n+m}{2}\phi_{\mathrm{RT}}}{\sin\frac{n+m}{2}\phi_{\mathrm{RT}}} < mn|T|00 > \tag{46}$$

Therefore, we find for the state $|v_1 >$:

$$|v_1 >= \sum_{mn} -i\frac{\cos\frac{n+m}{2}\phi_{\mathrm{RT}}}{\sin\frac{n+m}{2}\phi_{\mathrm{RT}}} < mn|T|00 > |mn > \tag{47}$$

This is a beautiful generalized result for the effect of thermal lensing in the SRC. In Section 6 we will use that for finding a an expression of mode-mismatch for a more realistic thermal noise resulting from Gaussian heating.

But before that we can cross-check the result against the ray transfer matrix calculation in Section 4 by noting that a parabolic thermal lens is given by

$$T = i\frac{kr^2}{2}\kappa \tag{48}$$

(see note [1]) and therefore, for the LG10 mode ($n + m = 2$), we find

$$< LG10|T|00 >= \frac{-iz_R}{2}\kappa \tag{49}$$

and therefore

$$|v_1 >= -\frac{\cos\phi_{\mathrm{RT}}}{\sin\phi_{\mathrm{RT}}}\frac{z_R}{2}\kappa|LG10 > \tag{50}$$

5.2 Mode-Matching at SRM

6 Summation for Thin Thermal Lens

We can expand a thin thermal lens heated with a Gaussian profile into higher order modes (see note [1]):

$$\begin{aligned} T|LG00\rangle =&|LG00\rangle - i\epsilon(|LG10\rangle + \frac{1}{4}|LG20\rangle + \frac{1}{12}|LG30\rangle + \frac{1}{32}|LG40\rangle + \frac{1}{80}|LG50\rangle \\ &+ \frac{1}{192}|LG60\rangle + \frac{1}{448}|LG70\rangle + \frac{1}{1024}|LG80\rangle + \frac{1}{2304}|LG90\rangle + ...) + O(\epsilon^2) \end{aligned} \tag{51}$$

with $\epsilon = \frac{z_R\kappa}{2}$ the strength of the fundamental lens. Explicitly

$$T|LG00\rangle = |LG00\rangle - i\frac{z_R\kappa}{2}\sum_p \frac{1}{2^{p-1}p}|LGp0\rangle + O(\left(\frac{z_R\kappa}{2}\right)^2) \tag{52}$$

where we also used the explicit expression for the prefactors - see for instance The On-Line Encyclopedia of Integer Sequences, OEIS A001787. We can sum over the coefficients in quadrature and get the total overlap power loss at the ITM:

$$< v_1|v_1 >= \left(\frac{z_R\kappa}{2}\right)^2 \sum_p \frac{1}{2^{2p-2}p^2}\frac{\cos^2 p\phi_{\mathrm{RT}}}{\sin^2 p\phi_{\mathrm{RT}}} \tag{53}$$

Similarly, we find for the total overlap power loss at the SRM:

$$< \tilde{v_1}|\tilde{v_1} >= \left(\frac{z_R\kappa}{2}\right)^2 \sum_p \frac{1}{2^{2p-2}p^2}\frac{1}{\sin^2 p\phi_{\mathrm{RT}}} \tag{54}$$

which is plotted in figure 2. Again, this is indeed a direct generalization of the result we received based on the ray transfer matrix approach for the lowest order mode.

7 Conclusion

Equation 53 and Figure 1 assume a clean, uniform absorption thermal lens without radiative cooling of the optic. However, these assumptions only weakly affect the relative contributions of higher order modes. The following main conclusion should not depend on those details:

Figure 1 suggests a one-way Gouy phase somewhere around 35 degree could significantly reduce the sensitivity of thermal mode mismatch, compared to Advanced LIGO with a Gouy phase around 22 degree.

However, if we have an actuator that can compensate the quadratic part of the lens - for example an optics displacement actuator - we can recover the loss from the purely quadratic lens, and are left will the other higher-order modes. The actually makes a Gouy phase around 20deg optimal.

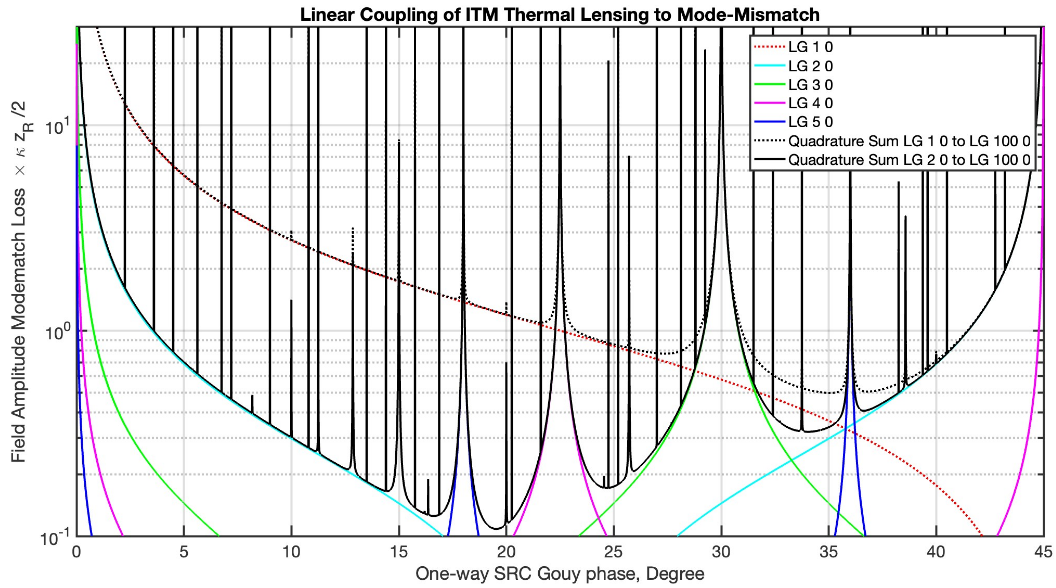

Figure 1: Field Amplitude Mode-Mismatch Loss at the ITM as function of the SRC Gouy phase. Uniform Gaussian heating from the interferometer beam is assumed, and no re-radiation losses are taken into account. The y axis show the 1st order coupling in units of $\kappa Z_R/2$, i.e. the square root of $< v_1 | v_1 >$ from equation 53.

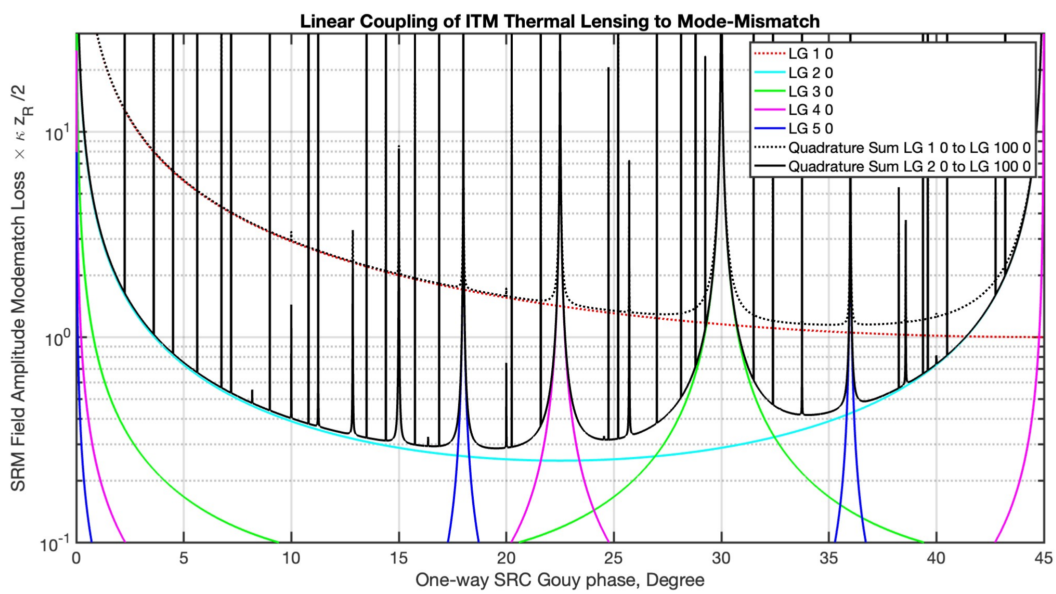

Figure 2: Field Amplitude Mode-Mismatch Loss at the SRM as function of the SRC Gouy phase. Uniform Gaussian heating from the interferometer beam is assumed, and no re-radiation losses are taken into account. The y axis show the 1st order coupling in units of $\kappa Z_R/2$, i.e. the square root of $< \tilde{v_1} | \tilde{v_1} >$ from equation 54.

A Appendix: Basics of Gaussian Beams

For completeness, here we summarize some properties of Gaussian beams. The field of a Gaussian beam is given by

$$\mathbf{\Psi} = \frac{A}{q} e^{-i\frac{k\vec{r}^2}{2q}} e^{-ikz} \tag{55}$$

Here, $q = z + iz_R$, with z the position along the beam axis, z_R the Rayleigh length, $\vec{r}$ the transverse beam position, $k = \frac{2\pi}{\lambda}$ the wave number, and A the field amplitude. For unit normalization, the amplitude is $A = \sqrt{\frac{kz_R}{\pi}}$.

This field can also be expressed as a superposition of plane waves:

$$\mathbf{\Psi} = \int d^2\xi \;\; \frac{-iA}{2\pi k} e^{i\frac{q\vec{\xi}^2}{2k}} e^{i\vec{\xi}\cdot\vec{r}} e^{-ikz} \tag{56}$$

Thus we can calculate the overlap between beams for any operator M in either basis:

$$\begin{aligned} \langle \mathbf{\Psi}_{q'} | M | \mathbf{\Psi}_q \rangle &= \int d^2r \left(\frac{A'^*}{q'^*} e^{i\frac{k\vec{r}^2}{2q'^*}} \right) M(r) \left(\frac{A}{q} e^{-i\frac{k\vec{r}^2}{2q}} \right) \\ &= \frac{k}{\pi} \frac{\sqrt{z'_R z_R}}{q'^* q} \int d^2r e^{-\frac{ik}{2}\left(\frac{q'^*-q}{q'^*q}\right)\vec{r}^2} M(r) \end{aligned} \tag{57}$$

$$\begin{aligned} \langle \mathbf{\Psi}_{q'} | M | \mathbf{\Psi}_q \rangle &= (2\pi)^2 \int d^2\xi \left(\frac{iA'^*}{2\pi k} e^{-i\frac{q'^*\vec{\xi}^2}{2k}} \right) M(\xi) \left(\frac{-iA}{2\pi k} e^{i\frac{q\vec{\xi}^2}{2k}} \right) \\ &= \frac{\sqrt{z'_R z_R}}{\pi k} \int d^2\xi e^{-\frac{-i}{2k}(q'^*-q)\vec{r}^2} M(\xi) \end{aligned} \tag{58}$$

In particular, for $M = 1$, the overlap of Gaussian beams with q and q' is

$$\langle \mathbf{\Psi}_{q'} | \mathbf{\Psi}_q \rangle = \frac{2i\sqrt{z'_R z_R}}{q - q'^*} \tag{59}$$

References

[1] Stefan W. Ballmer. Thermal Lensing - Some Back-of-the-Envelope Rules. Available at https://dcc.cosmicexplorer.org/CE-T2500014.