Thermal Lensing - Some Back-of-the-Envelope Rules

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Abstract

A note on thermal lensing basics that I should have posted a while ago.

1 Introduction

Cosmic Explorer DCC: CE-T2500014 LIGO DCC:LIGO-T2500239

2 Beam Overlap of two Gaussian Modes described by q-parameter

For two Gaussian Beams with beam parameters q_1 and q_2, given by equations 18 through 21, the field overlap is given by the exact expression

$$I_{1,2} = \langle \Psi_{q_1} | \Psi_{q_2} \rangle = \frac{2i\sqrt{(\Im q_1)(\Im q_2)}}{q_1 - q_2^*}, \tag{1}$$

where $\Im$ denotes the imaginary part. The power overlap then is $P_{1,2} = |I_{1,2}|^2$, or

$$P_{1,2} = 4\frac{(\Im q_1)(\Im q_2)}{|q_1 - q_2^*|^2}, \tag{2}$$

This expression assumes the two beans are co-aligned and co-centered. Note that $P_{1,2}$ is conserved under propagation through any optical system. In terms of inverse q-parameters $r_i = 1/q_i$, $P_{1,2}$ can also be written as

$$P_{1,2} = 4\frac{(\Im r_1)(\Im r_2)}{|r_2^* - r_1|^2}. \tag{3}$$

Indeed, if the two Gaussian beams are related via a lens of focal length f, $r2 = r1 + 1/f$, we find the exact expression

$$P_{1,2} = \frac{1}{1 + \left(\frac{\pi w^2}{\lambda}\frac{1}{2f}\right)^2}. \tag{4}$$

All four previous expressions are exact.

Alternatively, we can expand expression eq. 1 up to quadratic order if the two beams are similar:

$$P_{1,2} \approx 1 - \frac{|\Delta q|^2}{(2z_R)^2} \tag{5}$$

or

$$P_{1,2} \approx 1 - \left|\frac{\Delta z}{2z_R}\right|^2 - \left|\frac{\Delta w_0}{w_0}\right|^2 \tag{6}$$

We can also include any alignment offsets in this expansion:

$$P_{1,2} \approx 1 - \left|\frac{\Delta z}{2z_R}\right|^2 - \left|\frac{\Delta w_0}{w_0}\right|^2 - \left|\frac{\Delta x}{w_0}\right|^2 - \left|\frac{\Delta y}{w_0}\right|^2 - \left|\frac{\Delta x'}{\theta_0}\right|^2 - \left|\frac{\Delta y'}{\theta_0}\right|^2 \tag{7}$$

where Δx, Δy are the focus positions differences, $\Delta x'$, $\Delta y'$ the beam angle differences. $\theta_0 = \lambda/(\pi w_0)$ is the divergence angle.

3 Thin-Disk Lensing

For a uniformly heated thermal lens we have (see eq. 69)

$$\frac{1}{f} = [\beta + \bar{\alpha}(n-1)]\,\frac{I}{2\kappa} \tag{8}$$

in fact, to a good approximation, this condition is always true locally, i.e.

$$\frac{1}{f(x,y)} = [\beta + \bar{\alpha}(n-1)]\,\frac{I(x,y)}{2\kappa} \tag{9}$$

As a coarse approximation, we can assume that the average lens is approximately given by the mean intensity of the Gaussian beam, $P/(\pi w^2)$. In that approximation, the focal length is

$$\frac{1}{f} \approx [\beta + \bar{\alpha}(n-1)]\,\frac{P}{2\pi\kappa w^2} \tag{10}$$

and therefore, the power overlap is

$$P_{1,2} \approx \frac{1}{1 + \left([\beta + \bar{\alpha}(n-1)]\,\frac{P}{4\lambda\kappa}\right)^2} \approx 1 - [\beta + \bar{\alpha}(n-1)]^2 \left(\frac{P}{4\lambda\kappa}\right)^2 \tag{11}$$

Note that this is independent of the beam size w.

Indeed we can show that when we go beyond that coarse approximation, the exact first-order expression becomes

$$P_{1,2} \approx 1 - 1.0706 \cdot [\beta + \bar{\alpha}(n-1)]^2 \left(\frac{P}{4\lambda\kappa}\right)^2 \tag{12}$$

The lost power, however is no longer a Gaussian beam in a different basis.

We can also ask how well we can compensate this induced thermal lens with an ideal lens (quadratic phase profile), and how big that nes would be. The answer is that the coarse assumption above indeed gives the best-possible compensation lens

$$\frac{1}{f_{comp}} = -\,[\beta + \bar{\alpha}(n-1)]\,\frac{P}{2\pi\kappa w^2} \tag{13}$$

and that the residual power loss after compensation is given by

$$P_{1,2,comp} \approx 1 - 0.0706 \cdot [\beta + \bar{\alpha}(n-1)]^2 \left(\frac{P}{4\lambda\kappa}\right)^2 \tag{14}$$

We can also expand the lensed beam in the original Laguerre Gaussian basis:

$$[\mathrm{TL}]\,|LG00\rangle \;= \sqrt{(1 - 1.0706\epsilon^2)}|LG00\rangle + \epsilon|LG10\rangle + \frac{1}{4}\epsilon|LG20\rangle + \frac{1}{12}\epsilon|LG30\rangle + \frac{1}{150}\epsilon|\mathrm{higher}\rangle \tag{15}$$

without ideal lens compensation, or

$$[\mathrm{TL}]\,|LG00\rangle \;= \sqrt{(1 - 0.0706\epsilon^2)}|LG00\rangle + \frac{1}{4}\epsilon|LG20\rangle + \frac{1}{12}\epsilon|LG30\rangle + \frac{1}{150}\epsilon|\mathrm{higher}\rangle \tag{16}$$

with compensation. In these expressions [TL] represents the thermal noise induced by heating with the same Gaussian beam profile as the readout beam, and ϵ is given by

$$\epsilon = [\beta + \bar{\alpha}(n-1)]\left(\frac{P}{4\lambda\kappa}\right). \tag{17}$$

4 Appendix

4.1 Gaussian Beam Complex Beam Parameters

The complex beam parameter of a Gaussian beam with Rayleigh range z_R, at a distance z from its waist, is defined as

$$q = z + iz_R \ . \tag{18}$$

Beam size w and phase front radius of curvature R are then given by

$$\frac{1}{q} = \frac{1}{R} - i\frac{\lambda}{\pi w^2} \ , \tag{19}$$

where $\lambda = 2\pi/k$ is the wave length of the light. It allows expressing the Gaussian beam in a simple form:

$$\Psi(x, y, q) = A(x, y, q)e^{-ikz} \tag{20}$$

$$A(x, y, q) = \frac{A}{q}e^{-ik\frac{x^2+y^2}{2q}} \tag{21}$$

where A is a complex constant (amplitude). It can be helpful to introduce the field amplitude on the optical axis, $\psi = A/q$, which now evolves along the z-axis due to the Gouy phase evolution, but is unaffected when passing through a thin lens. Thus, for any given location on the optical axis z, the Gaussian beam is completely described by the two complex parameters ψ and q. The main advantage of this formalism becomes apparent when using ray-transfer matrices M defined in geometric optics (e.g. Saleh, Teich) to represent the action of a full optical system. The two complex parameters (q_f,ψ_f) after the system are given in terms of the initial parameters (q_i,ψ_i) by

$$M \begin{pmatrix} \frac{1}{\psi_i} \\ \\ \frac{1}{\psi_i q_i} \end{pmatrix} = \begin{pmatrix} \frac{1}{\psi_f} \\ \\ \frac{1}{\psi_f q_f} \end{pmatrix} \ , \tag{22}$$

and the change of the Gouy phase through the system, $\Delta\phi$, is given by

$$e^{i\Delta\phi} = \sqrt{\frac{\psi_f}{\psi_f^*}\frac{\psi_i^*}{\psi_i}} \ . \tag{23}$$

This expression is consistent with the usual definition of local Gouy phase for a Gaussian beam as $\phi = \arctan z/z_R$, but preserves the Gouy phase when propagating through a lens. To prove expressions 22 and 23 it is sufficient to verify them for a pure free-space propagation and a pure lens.

If we now introduce astigmatism, either intensionally with cylindrical lenses or accidentally through imperfections, cylindrical symmetry around the beam axis will be lost. As long as we introduce this astigmatism along a pre-determined axis (say the x-axis), we can simply proceed by introducing separate q-parameters for the x- and y-axis, q_x and q_y. Since ray-transfer matrices are introduced with only 1 transverse axis, the propagation of q_x and q_y is done with ray-transfer matrices defined for the corresponding transverse axis. Thus we now have a separately-defined beam size w_x, w_y, phase front radius of curvature R_x, R_y, Rayleigh range z_{Rx}, $z_R y$ and Gouy phase ϕ_x and ϕ_y for each of the two transverse directions. The corresponding fundamental Gaussian beam is given by

$$\Psi(x, y, q_x, q_y) = A(x, y, q_x, q_y)e^{-ikz} \tag{24}$$

$$A(x, y, q_x, q_y) = \frac{A}{\sqrt{q_x q_y}}e^{-ik\frac{x^2}{2q_x}}e^{-ik\frac{y^2}{2q_y}} \tag{25}$$

where A is again a complex amplitude. Next we introduce the Hermite-Gaussian basis set corresponding to the fundamental Gaussian beam. In the literature this is typically done only relative to a single q-parameter, but it directly generalizes to the case with separate q_x and q_y parameters:

$$\Psi_{nm}(x, y, q_x, q_y) = A_{nm}(x, y, q_x, q_y)e^{-ikz} \tag{26}$$

$$A_{nm}(x, y, q_x, q_y) = NA_n(x, q_x)A_m(y, q_y) \tag{27}$$

$$A_p(\xi, q_\xi) = e^{ip\phi_\xi}\sqrt{\frac{1}{2^p p!}\psi_\xi}\ \ H_p(\sqrt{2}\frac{\xi}{w_\xi})\ \ e^{-ik\frac{\xi^2}{2q_\xi}} \tag{28}$$

$$\psi_\xi = \sqrt{\frac{2}{\pi}}\ \frac{e^{i\phi_\xi}}{w_\xi} = \sqrt{\frac{2z_R}{\lambda}}\frac{i}{q_\xi} \tag{29}$$

$$H_0(\eta) = 1\ \ ,\ \ H_{p+1}(\eta) = 2\eta H_p(\eta) - \frac{d}{d\eta}H_p(\eta) \tag{30}$$

Here, we redefined the overall amplitude N such that the total power P in a mode is simply given by $P = \int |\Psi_{nm}|^2 dxdy = |N|^2$. That equations 24 and 25 are of the same form as equations 26 to 30 can be seen by using the identity $iz_R/q = e^{i\phi}w_0/w$. Furthermore we defined ψ_ξ in analog to the field amplitude ψ introduced after equation 21, that is the field amplitude on the optical axis of the fundamental mode. It thus evolves, together with q_ξ, according to equations 22 and 23. Note though that there is an extra Gouy phase term for the higher order modes that is explicitly excluded from the definition of ψ_ξ. As a result, the overall Gouy phase evolution of $\Psi_{nm}(x, y, q_x, q_y)$ is proportional to $e^{i(n+1/2)\phi_x + i(m+1/2)\phi_y}$.

As expected, these modes still solve the paraxial Helmholtz equation

$$(\triangle_T - 2ik\frac{\partial}{\partial z})A_{nm}(x, y, q_x, q_y) = 0 \tag{31}$$

exactly. Finally, in the main text we use the simplified bra-ket notation for readability:

$$|HG_{nm}\rangle = |\Psi_{nm}(x, y, q_x, q_y)\rangle. \tag{32}$$

4.2 Spatial Fourier Transform

4.2.1 Definition

We use a n-dimensional spatial Fourier transform to solve the heat diffusion equation. We define

$$f(\vec{x}) = \int d^n k f_k exp(i\vec{k}\vec{x}) \tag{33}$$

and

$$f_k = \frac{1}{(2\pi)^n}\int d^2 x f(\vec{x}) exp(-i\vec{k}\vec{x}) \tag{34}$$

Depending on the application, $n = 1$, $n = 2$ or $n = 3$ is used.

4.2.2 Parseval's Theorem

In this notation, Parseval's theorem becomes

$$\int d\vec{x}^n f(\vec{x})g(\vec{x}) = (2\pi)^n \int d\vec{k}^2 f^*_{\vec{k}} g_{\vec{k}} \tag{35}$$

4.2.3 Examples

For example, for a normalized field

$$\Psi(\vec{x}) = \sqrt{\frac{2}{\pi}}\frac{1}{w}e^{-\frac{\vec{x}^2}{w^2}} \tag{36}$$

the intensity profile is

$$I(\vec{x}) = \frac{2}{\pi}\frac{1}{w^2}e^{-\frac{2\vec{x}^2}{w^2}} \tag{37}$$

and fulfills $\int d\vec{x}^2 I(\vec{x}) = 1$. The Fourier transform of $I(\vec{x})$ thus is

$$I_k = \frac{1}{(2\pi)^2}e^{-\frac{\vec{k}^2 w^2}{8}} \tag{38}$$

As another example, we can calculate the field overlap of two Gaussian beams with spot size w across a thermal lens with phase profile $\phi(r) = -\frac{\pi}{\lambda f} r^2$.

$$I = \left\langle \Psi \left| e^{-i\frac{\pi}{\lambda f} r^2} \right| \Psi \right\rangle = \frac{2}{\pi w^2} \int d^2 r e^{-\left(\frac{2}{w^2} + i\frac{\pi}{\lambda f}\right) r^2} = \frac{1}{1 + i\frac{\pi w^2}{2\lambda f}} \tag{39}$$

in agreement with equation 4.

4.3 Thermal Diffusion

4.3.1 Diffusion Equation

The equation of thermal diffusion is given by

$$C\rho \dot{T} = \kappa \Delta T \tag{40}$$

The heat flow is given by

$$\vec{j} = -\kappa \vec{\nabla} T \tag{41}$$

with the boundary conditions , usually at $z = 0$, given by

$$I(x, y, z = 0) = -\kappa \nabla_z T(x, y, z = 0) \tag{42}$$

defining the incident net heat flow.

4.3.2 Thin, Stationary, Spherically Symmetric Thermal Lens

The optical thickness in z-direction of a thin, thermal lens with stationary heat flow (i.e. no time dependence in the temperature field) is given by

$$\triangle z(r) = [\beta + \bar{\alpha}(n-1)] \int_0^h T(r,z)dz = \alpha_{\text{eff}} \int_0^h T(r,z)dz \tag{43}$$

where $\beta = \partial n/\partial T$, $\bar{\alpha}$ is the transverse-constrained expansion coefficient, and $\alpha_{\text{eff}} = \beta + \bar{\alpha}(n-1)$.

Applying the 2-dim transverse Laplace to this equation we find

$$\Delta_2 \triangle z(r) = \alpha_{\text{eff}} \int_0^h \Delta_2 T(r,z)dz = -\alpha_{\text{eff}} \int_0^h \frac{\partial^2}{\partial z^2} T(r,z)dz = -\alpha_{\text{eff}} \left[\frac{\partial}{\partial z} T(r,z)\right]_0^h = -\frac{\alpha_{\text{eff}}}{\kappa} I(r) \tag{44}$$

We can now assume that the heating profile is given by the laser beam itself, i.e.

$$I(r) = P \frac{2}{\pi} \frac{1}{w^2} e^{-\frac{2r^2}{w^2}} \tag{45}$$

with P the total power deposited by the beam in the optic. Using the cylindrical form of the Laplace operator, we have

$$\Delta_2 \triangle z(r) = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} \triangle z(r) = -\frac{\alpha_{\text{eff}} P}{\kappa} \frac{2}{\pi} \frac{1}{w^2} e^{-\frac{2r^2}{w^2}} \tag{46}$$

or

$$r \Delta_2 \triangle z(r) = \frac{\partial}{\partial r} r \frac{\partial}{\partial r} \triangle z(r) = -\frac{\alpha_{\text{eff}} P}{\pi \kappa} \frac{2r}{w^2} e^{-\frac{2r^2}{w^2}} \tag{47}$$

Since we don't care about a constant offset in $\triangle z$, we can simply integrate from 0 to r, and divide the result by r:

$$\frac{\partial}{\partial r} \triangle z(r) = -\frac{\alpha_{\text{eff}} P}{\pi \kappa} \frac{1}{r} \left[-\frac{1}{2} e^{-\frac{2r^2}{w^2}}\right]_0^r = -\frac{\alpha_{\text{eff}} P}{2\pi\kappa} \frac{\left(1 - e^{-\frac{2r^2}{w^2}}\right)}{r} \tag{48}$$

We make the profile scale-independent by introducing $\tilde{r} = \sqrt{2}\frac{r}{w}$, and $\triangle z(r) = \frac{\alpha_{\text{eff}} P}{8\pi\kappa} \triangle \tilde{z}\left(\sqrt{2}\frac{r}{w}\right)$ with

$$\frac{\partial}{\partial \tilde{r}} \triangle \tilde{z}(\tilde{r}) = -4 \frac{\left(1 - e^{-\tilde{r}^2}\right)}{\tilde{r}} \tag{49}$$

(The factor 4 will make sense later). One more integration gives us $\triangle z(r)$. While there are closed expressions for this integral, it is easier to numerically integrate this.

The thermal lens phase profile is then given by

$$[\text{TL}] = e^{i\frac{2\pi}{\lambda}\triangle z(r)} = e^{i\frac{\alpha_{\text{eff}} P}{4\lambda\kappa}\triangle \tilde{z}(\tilde{r})} = e^{i\epsilon\triangle \tilde{z}(\tilde{r})} \tag{50}$$

where we set $\epsilon = \frac{\alpha_{\text{eff}} P}{4\lambda\kappa}$. The first four rotational-symmetric Laguerre-Gauss beam profiles are

$$|LG00\rangle = \sqrt{\frac{2}{\pi}} \frac{1}{w} e^{-\frac{r^2}{w^2}} = \sqrt{\frac{2}{\pi}} \frac{1}{w} e^{-\frac{\tilde{r}^2}{2}} \tag{51}$$

$$|LG10\rangle = |LG00\rangle \left[1 - \tilde{r}^2\right] \tag{52}$$

$$|LG20\rangle = |LG00\rangle \left[1 - 2\tilde{r}^2 + \frac{1}{2}\tilde{r}^4\right] \tag{53}$$

$$|LG30\rangle = |LG00\rangle \left[1 - 3\tilde{r}^2 + \frac{3}{2}\tilde{r}^4 - \frac{1}{6}\tilde{r}^6\right] \tag{54}$$

We can now find the scattering coefficients into the higher-order modes. We drop any pure phase, as this can be absorbed in the basis definition. We find to first order in $\epsilon = \frac{\alpha_{\text{eff}} P}{4\lambda\kappa}$ as

$$|\langle LG00|\,[\text{TL}]\,|LG00\rangle\,| \approx \sqrt{(1 - 1.0706\epsilon^2)} \tag{55}$$

$$|\langle LG10|\,[\text{TL}]\,|LG00\rangle\,| = \epsilon + O(\epsilon^2) \tag{56}$$

$$|\langle LG20|\,[\text{TL}]\,|LG00\rangle\,| = \frac{\epsilon}{4} + O(\epsilon^2) \tag{57}$$

$$|\langle LG30|\,[\text{TL}]\,|LG00\rangle\,| = \frac{\epsilon}{12} + O(\epsilon^2) \tag{58}$$

4.3.3 Thin Disk Thermal Diffusion

We can use the 2D spatial and 1D temporal Fourier transform and find

$$\left(\frac{i\Omega C\rho}{\kappa}+\vec{k}^2\right)T_k(z)=T_k''(z) \tag{59}$$

where the z-derivative of T is $\nabla_z T = T'$. We thus find the column integral of T as

$$\int_0^h T_k(z)dz=\frac{\kappa}{\left(i\Omega C\rho+\kappa\vec{k}^2\right)}\int_0^h T_k''(z)dz=\frac{\kappa T_k'|_{z=h}-\kappa T_k'|_{z=0}}{\left(i\Omega C\rho+\kappa\vec{k}^2\right)} \tag{60}$$

where h is the thickness, which can also be taken to be infinity, $h=\infty$. We will either ignore the heat flow on the back side I(x,y,z=h), or include it in the total $I(x,y)=I(x,y,z=0)+I(x,y,z=h)$, and find for the column integral of T:

$$\int_0^h T_k(z)dz=\frac{I_k}{\left(i\Omega C\rho+\kappa\vec{k}^2\right)} \tag{61}$$

This result is useful in two special cases. First, for moderately thin disks, thermal expansion and lensing only care about the the column integral of T. For instance, the change in optical thickness in z-direction is given by

$$\triangle z_k=[\beta+\bar{\alpha}(n-1)]\int_0^h T_k(z)dz=[\beta+\bar{\alpha}(n-1)]\frac{I_k}{\left(i\Omega C\rho+\kappa\vec{k}^2\right)} \tag{62}$$

where $\beta=\partial n/\partial T$ and $\bar{\alpha}$ is the effective expansion coefficient under the constraint that the material can only expand in z direction.

Second, for very thin thermal lenses, taking T as constant across the disk thickness is a good approximation. Thus we find

$$T_k=\frac{I_k}{h\left(i\Omega C\rho+\kappa\vec{k}^2\right)} \tag{63}$$

or, for the steady-state solution $\Omega=0$

$$\vec{k}^2T_k=\frac{I_k}{h\kappa} \tag{64}$$

This result might be easier without the 2D spatial Fourier transform:

$$\Delta_2T(x,y)=-\frac{I(x,y)}{h\kappa} \tag{65}$$

4.3.4 Thin Disk and Uniform Heating

For uniform heating across the surface of the thin disk, i.e. if $I(x,y)=I$ is constant, and we have cylindrical symmetry ($r^2=x^2+y^2$), the result is

$$T(r)=T_0-\frac{I}{4h\kappa}r^2 \tag{66}$$

that is a perfect quadratic lens. The corresponding optical thickness is

$$\triangle z=\triangle z_0-[\beta+\bar{\alpha}(n-1)]\frac{I}{4\kappa}r^2 \tag{67}$$

We can compare that to the optical thickness of a lens with focal length f:

$$\triangle z=\triangle z_0-\frac{1}{2f}r^2 \tag{68}$$

and find

$$\frac{1}{f}=[\beta+\bar{\alpha}(n-1)]\frac{I}{2\kappa} \tag{69}$$

5 Laguerre Expansion

Here we list some useful expansions of common operators in rotationally-symmetric LaGuerre modes.

5.1 Laguerre Modes

When studying rotationally symmetric effects (such as uniform thermal lensing), the LaGuerre basis for Gaussian beams is most useful. they can be defined as follows ($\tilde{r} = \frac{\sqrt{2}r}{w}$):

$$|LG00\rangle = \sqrt{\frac{2}{\pi}}\frac{1}{w}e^{-\frac{r^2}{w^2}} = \sqrt{\frac{2}{\pi}}\frac{1}{w}e^{-\frac{\tilde{r}^2}{2}} \tag{70}$$

$$|LGn0\rangle = |LG00\rangle \mathbf{laguerreL}(n, \tilde{r}^2) \tag{71}$$

where $\mathbf{laguerreL}(b, x)$ is the MATLAB notation for the LaGuerre polynomial:

$$\begin{aligned}
\mathbf{laguerreL}(n,x) &= \frac{\mathrm{e}^x}{n!}\frac{d^n}{dx^n}\left(x^n \mathrm{e}^{-x}\right) \\
\mathbf{laguerreL}(0,\tilde{r}^2) &= 1 \\
\mathbf{laguerreL}(1,\tilde{r}^2) &= 1 - \tilde{r}^2 \\
\mathbf{laguerreL}(2,\tilde{r}^2) &= 1 - 2\tilde{r}^2 + \frac{1}{2}\tilde{r}^4 \\
\mathbf{laguerreL}(3,\tilde{r}^2) &= 1 - 3\tilde{r}^2 + \frac{3}{2}\tilde{r}^4 - \frac{1}{6}\tilde{r}^6 \\
\mathbf{laguerreL}(4,\tilde{r}^2) &= 1 - 4\tilde{r}^2 + 3\tilde{r}^4 - \frac{2}{3}\tilde{r}^6 + \frac{1}{24}\tilde{r}^8 \\
\mathbf{laguerreL}(5,\tilde{r}^2) &= 1 - 5\tilde{r}^2 + 5\tilde{r}^4 - \frac{5}{3}\tilde{r}^6 + \frac{5}{24}\tilde{r}^8 - \frac{1}{120}\tilde{r}^{10}
\end{aligned} \tag{72}$$

5.2 Quadratic Lens

For a strictly paraboloid (quadratic) lens QL with focal length f, we define $g = \frac{\pi w^2}{2\lambda f}$. Its phase map is given by

$$QL = \mathrm{e}^{ig\tilde{r}^2} \tag{73}$$

and we find

$$QL|LG00\rangle = \mathrm{e}^{ig\tilde{r}^2}|LG00\rangle = \frac{1}{1-ig}\sum_{n=0}^{\infty}\left(\frac{-ig}{1-ig}\right)^n |LGn0\rangle \tag{74}$$

5.3 Thermal Lens

Similarly, we can look at the lens profile found in equation 49, $\triangle\tilde{z}(\tilde{r})$, and find the expansion

$$\begin{aligned}
\left(\triangle\tilde{z} + \ln(4)\right)|LG00\rangle = & |LG10\rangle + \frac{1}{4}|LG20\rangle + \frac{1}{12}|LG30\rangle + \frac{1}{32}|LG40\rangle + \frac{1}{80}|LG50\rangle \\
& + \frac{1}{192}|LG60\rangle + \frac{1}{448}|LG70\rangle + \frac{1}{1024}|LG80\rangle + \frac{1}{2304}|LG90\rangle
\end{aligned} \tag{75}$$

With the thermal lens defined via ($\epsilon = \frac{\alpha_{\mathrm{eff}}P}{4\lambda\kappa}$, and note that $\triangle\tilde{z}$ is negative)

$$TL = \mathrm{e}^{-i\epsilon\left(\triangle\tilde{z}+\ln(4)\right)} \tag{76}$$

we also find the expansion up to order ϵ:

$$TL|LG00\rangle = \mathrm{e}^{-i\epsilon\left(\triangle\tilde{z}+\ln(4)\right)}|LG00\rangle = |LG00\rangle - i\epsilon\left(|LG10\rangle + \frac{1}{4}|LG20\rangle + \frac{1}{12}|LG30\rangle + ...\right) + O(\epsilon^2) \tag{77}$$

5.4 Matrix elements for $\tilde{r}^{2n}$

Furthermore, it is useful to have the matrix elements of $\tilde{r}^{2n}$:

$$R_{kl}^{(n)} = \langle LGk0|\tilde{r}^{2n}|LGl0\rangle) \tag{78}$$

$$R_{kl}^{(1)} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -2 & 0 & 0 & 0 \\ 0 & -2 & 5 & -3 & 0 & 0 \\ 0 & 0 & -3 & 7 & -4 & 0 \\ 0 & 0 & 0 & -4 & 9 & -5 \\ 0 & 0 & 0 & 0 & -5 & 11 \end{pmatrix} \tag{79}$$

$$R_{kl}^{(2)} = \begin{pmatrix} 2 & -4 & 2 & 0 & 0 & 0 \\ -4 & 14 & -16 & 6 & 0 & 0 \\ 2 & -16 & 38 & -36 & 12 & 0 \\ 0 & 6 & -36 & 74 & -64 & 20 \\ 0 & 0 & 12 & -64 & 122 & -100 \\ 0 & 0 & 0 & 20 & -100 & 182 \end{pmatrix} \tag{80}$$

$$R_{kl}^{(3)} = \begin{pmatrix} 6 & -18 & 18 & -6 & 0 & 0 \\ -18 & 78 & -126 & 90 & -24 & 0 \\ 18 & -126 & 330 & -414 & 252 & -60 \\ -6 & 90 & -414 & 882 & -972 & 540 \\ 0 & -24 & 252 & -972 & 1854 & -1890 \\ 0 & 0 & -60 & 540 & -1890 & 3366 \end{pmatrix} \tag{81}$$

$$R_{kl}^{(4)} = \begin{pmatrix} 24 & -96 & 144 & -96 & 24 & 0 \\ -96 & 504 & -1056 & 1104 & -576 & 120 \\ 144 & -1056 & 3144 & -4896 & 4224 & -1920 \\ -96 & 1104 & -4896 & 11304 & -14976 & 11520 \\ 24 & -576 & 4224 & -14976 & 30024 & -36000 \\ 0 & 120 & -1920 & 11520 & -36000 & 66024 \end{pmatrix} \tag{82}$$

$$R_{kl}^{(5)} = \begin{pmatrix} 120 & -600 & 1200 & -1200 & 600 & -120 \\ -600 & 3720 & -9600 & 13200 & -10200 & 4200 \\ 1200 & -9600 & 32520 & -60600 & 67200 & -44400 \\ -1200 & 13200 & -60600 & 153720 & -237600 & 230400 \\ 600 & -10200 & 67200 & -237600 & 510120 & -700200 \\ -120 & 4200 & -44400 & 230400 & -700200 & 1350360 \end{pmatrix} \tag{83}$$