THE COSMIC EXPLORER GRAVITATIONAL WAVE OBSERVATORY

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Beamsplitter in a Strongly Convergent Telescope

Matthew Todd[a], Stefan Ballmer[b]
[a]*Syracuse University*, [b]*Syracuse University*

This is an internal working note of the COSMIC EXPLORER project.

1 Introduction

During design discussion for the Cosmic Explorer optical layout, the question arose whether it is acceptable to have a beamsplitter inside the strongly convergent telescope. Two separate issues were discussed:

- The reflectivity of the beamsplitter depends on the angle of incidence. Hence, for a strongly curved beam wavefront, the beamsplitter reflectivity varies across the beam. This is of particular concern for a large angle of incidence (e.g. 45 degrees), as the reflectivity will vary linearly with the angle of incidence.
- In a strongly curved wavefront the mode matching of x and y arm will become dependent on the macroscopic position of the BS.

2 BS Reflectivity Dependence on Angle of Incidence

2.1 Actual BS Coating Reflectivity

While the details of optical coatings are typically the manufacturer's intellectual property, we can model beamsplitter coatings as follows below. A MATLAB code following this principle is available in the Cosmic Explorer DCC: CE-T2300013[1].

- We use a dielectric stack with SiO_2 ($n_s = 1.45$ at $\lambda = 1064nm$) and Ta_2O_5 ($n_s = 2.06$ at $\lambda = 1064nm$).
- Snell's law dictates that in the beam is incident on the coating under an angle θ, then the propagation angles inside the coating layers are given by
$\sin(\theta) = n_s \sin(\theta_s) = n_c \sin(\theta_c)$.
- The top layer (SiO_2, $n_s = 1.45$) optical round trip has an extra minus sign from reflecting against vacuum ($n_v = 1$) on one side, and Ta_2O_5 ($n_c = 2.06$) on the other. It thus is designed to have $\lambda/2$ effective thickness: its normal-incident optical thickness is $d_{opt} = 0.5\lambda\cos(\theta_s)$, or physical thickness $d = 0.5\lambda\cos(\theta_s)/n_s$.
- All the other layers of SiO_2 and Ta_2O_5 are designed to have $\lambda/4$ effective thickness: their normal-incident optical thickness is $d_{opt} = 0.25\lambda\cos(\theta_i)$, or physical thickness $d = 0.25\lambda\cos(\theta_i)/n_i$.
- To build a coating with specified power reflectivity, we keep adding double-layers of SiO_2 and Ta_2O_5 until the reflectivity just exceeds the target power reflectivity (50% for our beamsplitter). Then we trim down the deepest layer of Ta_2O_5 until the power reflectivity exactly meets the target.

This procedure works for both S-polarization and P-polarization. But unless we are looking at normal incidence, it will produce different coatings for S- and P-polarization. See also Appendix B and the Cosmic Explorer DCC: CE-T2300013[1].

Figure 1 shows the amplitude reflectivity of a beamsplitter coating designed to be 50% power reflectivity for S-polarization under 45 degree angle of incident. This particular coating has the physical layer thicknesses d, the optical thicknesses d_{opt} and the index of refraction n as follows:

$$\begin{array}{rlcccccc}
d & = [& \text{vacuum}, & 320.3nm, & 121.3nm, & 160.2nm, & 43.9nm, & \text{substrate} \;] \\
d_{opt} & = [& \text{vacuum}, & 0.4365\lambda, & 0.2348\lambda, & 0.2183\lambda, & 0.0849\lambda, & \text{substrate} \;] \\
n & = [& 1, & 1.45, & 2.06, & 1.45, & 2.06, & 1.45 \;]
\end{array} \tag{1}$$

We can now expand the reflectivity to second order around the $\theta = \pi/4$ (i.e. 45 degree) point, which yields

$$r = r_0 \left(1 + f_1\alpha + \frac{f_2}{2}\alpha^2\right) \tag{2}$$

where $\alpha = \theta - \pi/4$. For this particular coating we find:

$$\begin{array}{ll}
f_1 & = +0.6320 - 1.4190i \quad \text{radian}^{-1} \\
f_2 & = -2.5025 - 1.3331i \quad \text{radian}^{-2}
\end{array} \tag{3}$$

While the exact values for f_1 and f_2 will vary for actual coatings, it is clear that they will typically be of order unity, with both real and imaginary parts. Also, third and higher-order terms will be negligible over the typical range of angles encountered.

2.2 Effect of Actual BS Reflectivity on Gaussian Beam

The quality of the quadratic fit in figure 1, suggests that the expansion (equation 2) should provide an accurate representation of the effect of the beamsplitter.

Since the reflectivity is directly a function of the angle of incidence, it is easier to quantify the effect on a Gaussian beam in the Fourier domain (see Appendix A).

Linear Term: The linear term in the reflectivity scatters the fundamental (00) mode into a (10) mode; that is, it is an effective beam deflection or displacement (depending on the complex sign). Since

$$|10> = -i\sqrt{\frac{2z_R}{k}}\xi_x|00> \tag{4}$$

with $\alpha = \frac{\xi_x}{k}$ the size of the effect is :

$$< 10|\frac{\xi_x}{k}|00> = \frac{i}{\sqrt{2z_Rk}} < 10|(-i)\sqrt{\frac{2z_R}{k}}\xi_x|00> \frac{i}{\sqrt{2z_Rk}} < 10|10> = \frac{i}{\sqrt{2z_Rk}} \tag{5}$$

and therefore

$$< 10|r|00> = r_0f_1 < 10|\frac{\xi_x}{k}|00> = r_0f_1\frac{i}{\sqrt{2z_Rk}} = r_0f_1\frac{i\theta_D}{2} \tag{6}$$

with $\theta_D = \sqrt{\frac{2}{z_Rk}}$ the Gaussian beam divergence angle. Thus, the size of the field scattering effect is $O\left(f_1\theta_D\right)$.

Also note that

$$< 00|\frac{\xi_x}{k}|00> = 0 \tag{7}$$

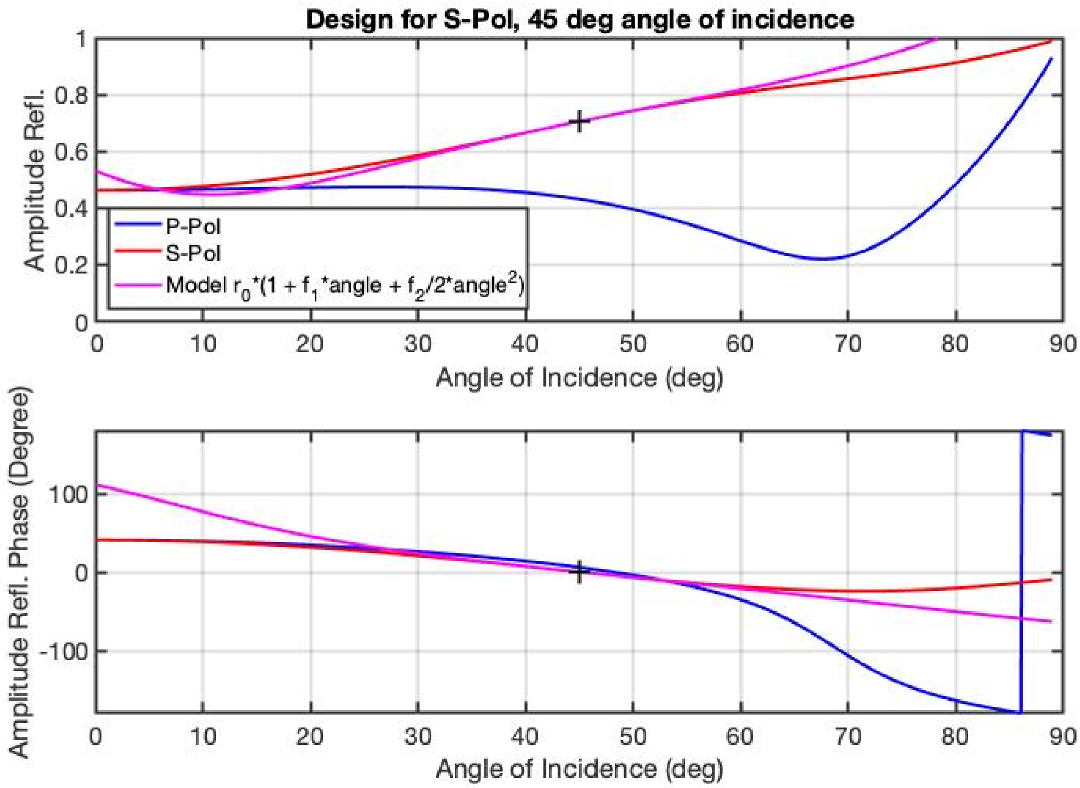

Figure 1: Amplitude reflectivity of a beamsplitter coating designed to be 50% power reflectivity for S-polarization under 45 degree. Shown are S-polarization amplitude reflectivity and phase in red (rotated to be 0 degree phase at 45 degree angle), P-polarization amplitude reflectivity and phase in blue, as well as a quadratic fit to the S-polarization amplitude reflectivity in magenta.

since we are scattering into a mode that is orthogonal to the fundamental.

Quadratic Term: Since equation 2 is used around $\theta = \pi/4$ (i.e. 45 degree), the quadratic term only depends on one direction, so we again have $\alpha = \frac{\xi_x}{k}$. We have

$$< 00|\frac{\xi_x^2}{k^2}|00 >= \frac{1}{2z_R k} \tag{8}$$

and thus

$$< 00|r|00 >= r_0 \left(1 + \frac{f_2}{2} < 00|\frac{\xi_x^2}{k^2}|00 >\right) = r_0 \left(1 + \frac{f_2}{4z_R k}\right) = r_0 \left(1 + \frac{f_2 \theta_D^2}{8}\right) \tag{9}$$

Thus, the size of the field scattering effect is $O\left(f_2 \theta_D^2\right)$, with $\theta_D = \sqrt{\frac{2}{z_R k}}$. The power scatter loss is thus approximately $\frac{f_2 \theta_D^2}{4}$.

In conclusion, we find

- The coefficients f_1 and f_2 in equation 2 are typically of order unity, but could be tuned to be intentionally small in a special coating design.
- The linear effect is order $O\left(f_1 \theta_D\right)$, but corresponds to a simple alignment change.
- The quadratic effect is order $O\left(f_2 \theta_D^2\right)$, and would cause a true mode-matching loss.
- In a telescope, the divergence angle will be approximately given by the beam spot radius w in the arm (12cm), over the telescope length. Thus, for example, for a 12m telescope, $\theta_D = 10^{-2}$, the scattered power loss is 25ppm $* f_2$.

3 BS Position Effect

If a beam splinter placed in a strongly converging telesope moves by a macroscopic amount it effectively changes the overlap of the two beams being combined. The effect is calculated in document CE-T2300011 [2] to be

$$\langle \mathbf{\Psi}_{q'} | \mathbf{\Psi}_q \rangle = \frac{1}{1 + i\frac{dx}{zR}} \tag{10}$$

where dx is the displacement of the beamsplitter and z_R is the Rayleigh length. Thus, for the power overlap we find

$$|\langle \mathbf{\Psi}_{q'} | \mathbf{\Psi}_q \rangle|^2 = \frac{1}{1 + \left(\frac{dx}{zR}\right)^2} \tag{11}$$

In words, what matters is directly the Rayleigh length of the beam passing through the BS. For example, if we assume a BS placement tolerance of 5mm, and a maximum acceptable loss of 25ppm, the minimum acceptable Rayleigh length is

$$z_{R,min} = \frac{5\text{mm}}{\sqrt{25 \cdot 10^{-6}}} = 1\text{m} \tag{12}$$

Note that, as mentioned above, a 12m telescope would have a divergence angle of approximately 12m telescope, $\theta_D = 10^{-2}$, which corresponds to a Rayleigh length of $z_R = \frac{2}{\theta_D^2 k} \approx$ 3.4mm. A maximum acceptable loss of 25ppm would thus require a BS placement with 17 micrometer precision.

4 Conclusion

Investigating both the beamsplitter reflectivity's dependence on the angle of incidence and the effect of beamsplitter misplacement on arm mode matching, we conclude that **mode matching** has the much bigger effect on interferometer design, and likely will be the limiting constraint for acceptable divergence angles through the beamsplitter.

A Appendix: Basics of Gaussian Beams

For completeness, here we summarize some properties of Gaussian beams. The field of a Gaussian beam is given by

$$\mathbf{\Psi} = \frac{A}{q} e^{-i\frac{k\vec{r}^2}{2q}} e^{-ikz} \tag{13}$$

Here, $q = z + iz_R$, with z the position along the beam axis, z_R the Rayleigh length, $\vec{r}$ the transverse beam position, $k = \frac{2\pi}{\lambda}$ the wave number, and A the field amplitude. For unit normalization, the amplitude is $A = \sqrt{\frac{kz_R}{\pi}}$.

This field can also be expressed as a superposition of plane waves:

$$\mathbf{\Psi} = \int d^2\xi \quad \frac{-iA}{2\pi k} e^{i\frac{q\vec{\xi}^2}{2k}} e^{i\vec{\xi}\cdot\vec{r}} e^{-ikz} \tag{14}$$

Thus we can calculate the overlap between beams for any operator M in either basis:

$$\begin{aligned} \langle \mathbf{\Psi}_{q'} | M | \mathbf{\Psi}_q \rangle &= \int d^2r \left(\frac{A'^*}{q'^*} e^{i\frac{k\vec{r}^2}{2q'^*}} \right) M(r) \left(\frac{A}{q} e^{-i\frac{k\vec{r}^2}{2q}} \right) \\ &= \frac{k}{\pi} \frac{\sqrt{z'_R z_R}}{q'^* q} \int d^2r e^{-\frac{ik}{2}\left(\frac{q'^*-q}{q'^*q}\right)\vec{r}^2} M(r) \end{aligned} \tag{15}$$

$$\begin{aligned} \langle \mathbf{\Psi}_{q'} | M | \mathbf{\Psi}_q \rangle &= (2\pi)^2 \int d^2\xi \left(\frac{iA'^*}{2\pi k} e^{-i\frac{q'^*\vec{\xi}^2}{2k}} \right) M(\xi) \left(\frac{-iA}{2\pi k} e^{i\frac{q\vec{\xi}^2}{2k}} \right) \\ &= \frac{\sqrt{z'_R z_R}}{\pi k} \int d^2\xi e^{-\frac{-i}{2k}(q'^*-q)\vec{r}^2} M(\xi) \end{aligned} \tag{16}$$

In particular, for $M = 1$, the overlap of Gaussian beams with q and q' is

$$\langle \mathbf{\Psi}_{q'} | \mathbf{\Psi}_q \rangle = \frac{2i\sqrt{z'_R z_R}}{q - q'^*} \tag{17}$$

B Appendix: Transfer Matrix for Coatings

For S-polarization, the amplitude reflectivity and transmissivity are given by (incident light traveling from first to second index):

$$r_{12}^S = \frac{n_1 \cos(\theta_1) - n_2 \cos(\theta_2)}{n_1 \cos(\theta_1) + n_2 \cos(\theta_2)} \tag{18}$$

$$t_{12}^S = \frac{2n_1 \cos(\theta_1)}{n_1 \cos(\theta_1) + n_2 \cos(\theta_2)} \tag{19}$$

Thus, the transfer matrix, taking a 2-dim vector of right- and left-propagating field from medium 1 to medium 2 is

$$T_{12}^S = \frac{1}{2n_2 \cos(\theta_2)} \begin{pmatrix} +n_1 \cos(\theta_1) + n_2 \cos(\theta_2) & -n_1 \cos(\theta_1) + n_2 \cos(\theta_2) \\ -n_1 \cos(\theta_1) + n_2 \cos(\theta_2) & +n_1 \cos(\theta_1) + n_2 \cos(\theta_2) \end{pmatrix} \tag{20}$$

Similarly, for P-polarization, reflectivity, transmitivity and the transfer matrix are given by:

$$r_{12}^P = \frac{n_1 \cos(\theta_2) - n_2 \cos(\theta_1)}{n_1 \cos(\theta_2) + n_2 \cos(\theta_1)} \tag{21}$$

$$t_{12}^P = \frac{2n_1 \cos(\theta_1)}{n_1 \cos(\theta_2) + n_2 \cos(\theta_1)} \tag{22}$$

$$T_{12}^P = \frac{1}{2n_2 \cos(\theta_2)} \begin{pmatrix} +n_1 \cos(\theta_2) + n_2 \cos(\theta_1) & -n_1 \cos(\theta_2) + n_2 \cos(\theta_1) \\ -n_1 \cos(\theta_2) + n_2 \cos(\theta_1) & +n_1 \cos(\theta_2) + n_2 \cos(\theta_1) \end{pmatrix} \tag{23}$$

More details on transfer matrices for coating reflectivity calculations can be found in Appendix A of the paper "Photothermal transfer function of dielectric mirrors for precision measurements" [3].

References

[1] Stefan W. Ballmer. MATLAB code for calculating coating reflectivities. Available at https://dcc.cosmicexplorer.org/CE-T2300013.

[2] Stefan W. Ballmer. Beam splitter placement error impact on mode overlap. Available at https://dcc.cosmicexplorer.org/CE-T2300011.

[3] Stefan W. Ballmer. Photothermal transfer function of dielectric mirrors for precision measurements. *Phys. Rev. D*, 91:023010, Jan 2015.